Critical behaviour of time-dependent Green functions of boson gas from renormalization group II

Juha Honkonen

July 31, 2019

Outline

Temporal regularization

Generating functional

Physics from generating function

Effective IR model

Renormalization and renormalization group

Functional representation of generating function

All GF@FT contained in the generating function(al)

$$G(A) = \mathrm{Tr}\left\{\hat{\rho}\, T \exp\left[A\hat{\varphi}_H\right]\right\} = \frac{Z_0}{Z_G}\Bigg\{\exp\Bigg(\frac{1}{2}\sum_{l=1}^{4}\frac{\delta}{\delta\varphi_l}\Delta_{ll}\frac{\delta}{\delta\varphi_l}$$

$$+\sum_{k<l}\frac{\delta}{\delta\varphi_k}n_{kl}\frac{\delta}{\delta\varphi_l}+\sum_{k,l=1}^{4}\frac{\delta}{\delta\varphi_k}d_{kl}\frac{\delta}{\delta\varphi_l}\Bigg)\exp\Bigg[\frac{1}{\hbar}S_V(\hbar\beta,0,\varphi_1)$$

$$-\frac{i}{\hbar}S_V(t_f,t_0,\varphi_2)+\frac{i}{\hbar}S_V(t_f,t_i,\varphi_3)-\frac{i}{\hbar}S_V(t_0,t_i,\varphi_4)+A\varphi_3\Bigg]\Bigg\}\Bigg|_{\varphi_i=0},$$

Standard Feynman rules produced by functional derivatives. Left side independent of time parameters $t_f > t_0 > t_i$, but they show in perturbation expansion generated by the right side.

Pinch singularities of self-energy

Pinch singularities are generated by expansion of full propagators in terms of self-energy:

$$D = \underline{\Delta} + \underline{\Delta}\Sigma\underline{\Delta} + \underline{\Delta}\Sigma\underline{\Delta}\Sigma\underline{\Delta} + \dots$$

Each correction term contains products of δ functions in frequency with coinciding arguments.

Formal inverse suggests a solution (Dyson equation)

$$D^{-1} = \underline{\Delta}^{-1} - \Sigma\,.$$

Self energy graphs are one-irreducible: no products of identical propagators.

Undamped oscillations

Ordinary quantum-mechanical oscillations are integrable in the momentum space (but this is very inconvenient in perturbation theory), since

$$\int_{-\infty}^{\infty} dk \, \exp\left(-i\,\frac{\hbar k^2}{2m}\right) \propto \int_0^{\infty} \frac{d\epsilon}{\sqrt{\epsilon}} \exp\left(-i\,\frac{\epsilon}{\hbar}\right) .$$

In loop integrals of GF@FT undamped oscillations occur, e.g.

$$-i\frac{\hbar k^2}{2m} + i\frac{\hbar(\mathbf{q}-\mathbf{k})^2}{2m} = \frac{i}{2m\hbar}\left(q^2 - 2\mathbf{k}\cdot\mathbf{q}\right)$$

exponential of which is not integrable over $\mathbf{k}$.

Zero-temperature GF do not have this problem: contractions either retarded or advanced, no pure oscillations.

Regularization by attenuation

To make each separate Feynman diagram finite introduce attenuation in both time directions and define the regularized propagators

$$\Delta_{\rm reg}(t,t';\mathbf{k}) = \theta(t-t')\exp\left[-i\omega(\mathbf{k})(t-t') - \gamma(t-t')\right] ,$$
$$\widetilde{\Delta}_{\rm reg}(t,t';\mathbf{k}) = \theta(t'-t)\exp\left[-i\omega(\mathbf{k})(t-t') + \gamma(t-t')\right] ,$$

$$d_{DD\,\rm reg}(t,t';\mathbf{k}) = \exp\left[-i\omega(\mathbf{k})(t-t') - \gamma|t-t'|\right]\overline{n}(\mathbf{k}) ,$$
$$d_{DE\,\rm reg}(t,t';\mathbf{k}) = \exp\left[\omega(\mathbf{k})(-i(t-t_0) + t' - \gamma|t-t_0|)\right]\overline{n}(\mathbf{k}) ,$$
$$d_{ED\,\rm reg}(t,t';\mathbf{k}) = \exp\left[\omega(\mathbf{k})(-t + i(t'-t_0) - \gamma|t'-t_0|)\right]\overline{n}(\mathbf{k}) .$$

Regularized simple contractions by substitution $\overline{n} \to 1$.

Regularization in Fourier space

In Fourier space, regularization removes δ functions in frequency

$$\Delta(\omega,\mathbf{k}) = \frac{i}{\omega-\omega(\mathbf{k})+i\varepsilon}\,, \qquad \widetilde{\Delta}(\omega,\mathbf{k}) = \frac{-i}{\omega-\omega(\mathbf{k})-i\varepsilon}\,,$$
$$d_{DD}(\omega,\mathbf{k}) = 2\pi\,\delta\left[\omega-\omega(\mathbf{k})\right]\overline{n}_{\mathbf{k}}\,, \qquad n_{DD}(\omega,\mathbf{k}) = 2\pi\,\delta\left[\omega-\omega(\mathbf{k})\right]\,.$$

by replacing $\varepsilon\to 0+$ by finite $\gamma>0$ in the Sokhotsky identity

$$\frac{1}{x\pm i\varepsilon} = P\,\frac{1}{x}\mp i\pi\delta(x) \longrightarrow \pi\delta(x) = \left(\frac{i}{x+i\varepsilon}-\frac{i}{x-i\varepsilon}\right) = \frac{2\varepsilon}{x^2+\varepsilon^2}\,,\ \text{e.g.}$$

$$\Delta_{\text{reg}}(\omega,\mathbf{k}) = \frac{i}{\omega-\omega(\mathbf{k})+i\gamma}\,, \qquad d_{DD\,\text{reg}}(\omega,\mathbf{k}) = \frac{4\gamma\,\overline{n}_{\mathbf{k}}}{\left[\omega-\omega(\mathbf{k})\right]^2+\gamma^2}\,.$$

Time limits in regularized theory

Regularized propagators allow safe pass to the limit $t_i \to -\infty$ and $t_f \to \infty$.

The (simplest) choice $t_0 \to -\infty$ makes $d_{DE\,\mathrm{reg}}$, $d_{ED\,\mathrm{reg}}$, $n_{DE\,\mathrm{reg}}$ and $n_{ED\,\mathrm{reg}}$ vanish and also removes the contribution of the fields ψ_2 and ψ_2^+ in interaction functional.

Three pairs of fields survive with the propagator

$$\underline{\Delta}_{\mathrm{reg}} = \begin{pmatrix} \Delta_{\mathrm{reg}} + d_{DD\,\mathrm{reg}} & 0 & n_{DD\,\mathrm{reg}} + d_{DD\,\mathrm{reg}} \\ 0 & \Delta_E + d_{EE} & 0 \\ d_{DD\,\mathrm{reg}} & 0 & \widetilde{\Delta}_{\mathrm{reg}} + d_{DD\,\mathrm{reg}} \end{pmatrix} .$$

The pair ψ_3^+ and ψ_3 (giving the grand partition function Z_G) is decoupled from the others. Let alone this, the only temperature dependence remains in the thermal contraction $d_{DD\,\mathrm{reg}}$.

Keldysh field theory

We are left with n-point functions in terms of two field pairs

$$G_n(x_1, x_2, \ldots, x_n) = \left\{ \exp\left(\sum_{l=1}^{2} \frac{\delta}{\delta \psi_l} \underline{\Delta}_{\mathrm{reg}\, ll} \frac{\delta}{\delta \psi_l^+} \right) \psi_1(x_1) \cdots \psi_1^+(x_n) \right.$$
$$\left. \times \exp\left[-\frac{ig}{2\hbar} \int_{-\infty}^{\infty} dt \int d\mathbf{x}\, \psi_1^{+2} \psi_1^2 + \frac{ig}{2\hbar} \int_{-\infty}^{\infty} dt \int d\mathbf{x}\, \psi_2^{+2} \psi_2^2 \right] \right\} \Bigg|_{\substack{\psi_i = 0 \\ \psi_i^+ = 0}},$$

with the regularized propagator

$$\underline{\Delta}_{\mathrm{reg}} = \begin{pmatrix} \Delta_{\mathrm{reg}} + d_{DD\,\mathrm{reg}} & n_{DD\,\mathrm{reg}} + d_{DD\,\mathrm{reg}} \\ d_{DD\,\mathrm{reg}} & \widetilde{\Delta}_{\mathrm{reg}} + d_{DD\,\mathrm{reg}} \end{pmatrix}.$$

To build perturbation theory and analyze divergences another set of Green functions (Keldysh 1965) is more convenient.

Propagator in RAK variables

Introduce RAK variables (Retarded-Advanced-Keldysh)

$$\begin{pmatrix} \eta \\ \eta^+ \\ \xi \\ \xi^+ \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_1 - \psi_2 \\ \psi_1^+ - \psi_2^+ \\ \psi_2 + \psi_1 \\ \psi_2^+ + \psi_1^+ \end{pmatrix} .$$

The reduction operator of the perturbation expansion becomes

$$P = \exp\left[-\frac{\delta}{\delta\eta} \widetilde{\Delta}_{\mathrm{reg}} \frac{\delta}{\delta\xi^+} + \frac{\delta}{\delta\xi} \Delta_{\mathrm{reg}} \frac{\delta}{\delta\eta^+} + \frac{\delta}{\delta\xi} \Delta^K_{\mathrm{reg}} \frac{\delta}{\delta\xi^+} \right] ,$$

where Δ^K_{reg} is the (regularized) **Keldysh function**

$$\Delta^K_{\mathrm{reg}}(t,t';\mathbf{k}) = \exp\left[\omega(\mathbf{k})(-i(t-t') - \gamma|t-t'|)\right]\left[1 + 2\overline{n}(\mathbf{k})\right] .$$

Interaction functional in RAK variables

The interaction functional becomes

$$S_V(\eta,\eta^+,\xi,\xi^+) = -\frac{g}{2}\int_{-\infty}^{\infty} dt \int d\mathbf{x}\, \left(\psi_1^{+2}\psi_1^2 - \psi_2^{+2}\psi_2^2\right)$$

$$= -\frac{g}{2}\int_{-\infty}^{\infty} dt \int d\mathbf{x}\, \left(\eta^+\xi^+\xi^2 + \xi^{+2}\xi\eta + \xi^+\eta^+\eta^2 + \eta^{+2}\eta\xi\right)$$

$$\equiv V_1 + V_2 + V_3 + V_4\,.$$

Separation of advanced and retarded propagators from (damped) oscillations brings about closed loops of identically directed temporal step functions which vanish.

This is similar to the loop theorem in the field theory of stochastic dynamics.

Perturbation theory in RAK variables

Features of reduction operator and interaction functional in RAK variables

$$P = \exp\left[-\frac{\delta}{\delta\eta}\widetilde{\Delta}_{\rm reg}\frac{\delta}{\delta\xi^+} + \frac{\delta}{\delta\xi}\Delta_{\rm reg}\frac{\delta}{\delta\eta^+} + \frac{\delta}{\delta\xi}\Delta^K_{\rm reg}\frac{\delta}{\delta\xi^+}\right],$$

$$S_V = -\frac{g}{2}\int dt \int d\mathbf{x}\,\left(\eta^+\xi^+\xi^2 + {\xi^+}^2\xi\eta + \xi^+\eta^+\eta^2 + {\eta^+}^2\eta\xi\right)\ :$$

- Only $\widetilde{\Delta}_{\rm reg}$ and $\Delta_{\rm reg}$ attached to field arguments η and η^+ in S_V.
- In $\widetilde{\Delta}_{\rm reg}$ and $\Delta_{\rm reg}$ the lesser time argument is always that of η or η^+.
- Vertices of S_V contain at least one η or η^+: outgoing $\widetilde{\Delta}_{\rm reg}$ or $\Delta_{\rm reg}$ started.
- Vertices of S_V contain at least one ξ^+ or ξ: incoming $\widetilde{\Delta}_{\rm reg}$ or $\Delta_{\rm reg}$ may be attached.

Loop theorem

Connected graph there with $\widetilde{\Delta}_{\mathrm{reg}}$ or Δ_{reg} between two vertices: necessarily contains a chain of equally directed temporal step functions.

Two possibilities: the chain goes through the graph or forms a closed loop which vanishes.

Consequences:

- ☐ One-irreducible graphs with external ξ or ξ^{+} arguments only vanish identically: bound to have closed loops of successive step functions in time. In particular,
- ☐ $\Gamma_{\xi\xi^{+}} = 0$: the field structure of the propagator matrix is preserved.
- ☐ $\Gamma_{\xi\xi\xi^{+}\xi^{+}} = 0$ – a limitation to generation of vertex structures under renormalization.

Functional integral for reduction operator

The standard trick to construct a functional integral relies on the identity

$$\exp\left[-\frac{\delta}{\delta\eta}\widetilde{\Delta}_{\rm reg}\frac{\delta}{\delta\xi^+}+\frac{\delta}{\delta\xi}\Delta_{\rm reg}\frac{\delta}{\delta\eta^+}+\frac{\delta}{\delta\xi}\Delta^K_{\rm reg}\frac{\delta}{\delta\xi^+}\right]$$

$$=\det\left|\frac{\left[\partial_t+i\omega(\mathbf{k})-\gamma\right]\left[\partial_t+i\omega(\mathbf{k})+\gamma\right]}{(2\pi)^2}\right|\int\mathcal{D}E\int\mathcal{D}E^+\int\mathcal{D}X\int\mathcal{D}X^+$$

$$\times\exp\left(-X^+\left[\frac{\partial}{\partial t}+i\omega(\mathbf{k})-\gamma\right]E-E^+\left[\frac{\partial}{\partial t}+i\omega(\mathbf{k})+\gamma\right]X\right.$$

$$\left.-E^+\left\{2\gamma\left[1+2\overline{n}(\mathbf{k})\right]\right\}E+E\frac{\delta}{\delta\eta}+E^+\frac{\delta}{\delta\eta^+}+X\frac{\delta}{\delta\xi}+X^+\frac{\delta}{\delta\xi^+}\right).$$

Linear exponentials of derivatives produce shifts in the integration variables.

Generating function in RAK variables

Functional-derivative representation brings about a functional integral

$$G(A,A^+,B,B^+) = \exp\left[-\frac{\delta}{\delta\eta}\widetilde{\Delta}_{\rm reg}\frac{\delta}{\delta\xi^+} + \frac{\delta}{\delta\xi}\Delta_{\rm reg}\frac{\delta}{\delta\eta^+} + \frac{\delta}{\delta\xi}\Delta^K_{\rm reg}\frac{\delta}{\delta\xi^+}\right]$$
$$\times \exp\left[\frac{i}{\hbar}S_V(\eta^\pm,\xi^\pm) + A^+\xi + +B^+\eta + A\xi^+ + B\eta^+\right]\Bigg|_{\eta^\pm=\xi^\pm=0}$$
$$= \int\mathcal{D}\eta\int\mathcal{D}\eta^+\int\mathcal{D}\xi\int\mathcal{D}\xi^+\exp\Bigg(-\xi^+\left[\frac{\partial}{\partial t} + i\omega(\mathbf{k}) - \gamma\right]\eta$$
$$-\eta^+\left[\frac{\partial}{\partial t} + i\omega(\mathbf{k}) + \gamma\right]\xi - \eta^+\left\{2\gamma\left[1+2\overline{n}(\mathbf{k})\right]\right\}\eta + \frac{i}{\hbar}S_V(\eta^\pm,\xi^\pm) + \text{sources}\Bigg)$$

Note that this is unambiguous in the regularized model only, when $\gamma > 0$ and $\Delta^K_{\rm reg}$ is not a solution of the free-field equation of motion.

Ambiguity of the functional integral

Unambiguous generating function of GF@FT:

$$G(J,J^+) = \int \mathcal{D}\psi_2 \mathcal{D}\psi_2^+ \mathcal{D}\psi_1 \mathcal{D}\psi_1^+ \exp\Big(J^+\psi_2 + J\psi_2^+ \\ + \psi_1^+ \left\{ \frac{\partial}{\partial t} + i\omega(\mathbf{k}) - \gamma\left[1 + 2\overline{n}(\mathbf{k})\right] \right\} \psi_1 + 2\psi_1^+ \gamma \left[1 + \overline{n}(\mathbf{k})\right] \psi_2 \\ - \psi_2^+ \left\{ \frac{\partial}{\partial t} + i\omega(\mathbf{k}) + \gamma\left[1 + 2\overline{n}(\mathbf{k})\right] \right\} \psi_2 + 2\psi_2^+ \gamma \overline{n}(\mathbf{k}) \psi_1 \\ + \frac{ig}{2\hbar} \left({\psi_1^+}^2 \psi_1^2 - {\psi_2^+}^2 \psi_2^2 \right) \Big) .$$

Lift regularization: $\gamma \to 0$

- ☐ Physical fields $\psi_2^\pm$ and auxiliary fields $\psi_1^\pm$ decouple
- ☐ Dependence on $\overline{n}(\mathbf{k})$ vanishes!

Action functional for generating function

The action functional of the regularized model

$$S = \xi^+ \left[i\hbar \frac{\partial}{\partial t} + \hbar\omega(\mathbf{k}) - i\hbar\gamma \right] \eta - \eta^+ \left[i\hbar \frac{\partial}{\partial t} + \hbar\omega(\mathbf{k}) + i\hbar\gamma \right] \xi$$
$$+ i\eta^+ \left\{ 2\hbar\gamma \left[1 + 2\overline{n}(\mathbf{k}) \right] \right\} \eta - \frac{g}{2} \left(\eta^+ \xi^+ \xi^2 + {\xi^+}^2 \xi\eta + \xi^+ \eta^+ \eta^2 + {\eta^+}^2 \eta\xi \right)$$

is the starting point for construction and renormalization of effective large-scale dynamic model for boson gas.

To keep perturbation theory tractable, the attenuation parameter γ must be dealt with after performing all spatial and temporal integrals of PT.

Propagators in dimensionless variables

Propagators of the effective IR theory in dimensionless variables

$$
\begin{aligned}
\langle \xi(t,\mathbf{k})\xi^+(t',-\mathbf{k})\rangle_0 &= \Delta^K_{\mathrm{IR\,reg}}(t,\mathbf{k}) = \frac{2}{k^2} e^{-iuk^2(t-t')-\alpha k^2|t-t'|}, \\
\langle \eta(t,\mathbf{k})\eta^+(t',-\mathbf{k})\rangle_0 &= 0, \\
\langle \xi(t,\mathbf{k})\eta^+(t',-\mathbf{k})\rangle_0 &= \Delta_{\mathrm{reg}}(t,\mathbf{k}) \\
&= \theta(t-t') e^{-iuk^2(t-t')-\alpha k^2|t-t'|}, \\
\langle \eta(t,\mathbf{k})\xi^+(t',-\mathbf{k})\rangle_0 &= -\widetilde{\Delta}_{\mathrm{reg}}(t,\mathbf{k}) \\
&= -\theta(t'-t) e^{-iuk^2(t-t')-\alpha k^2|t-t'|},
\end{aligned}
$$

with the dimensionless parameters α and u.

Basic action in dimensionless variables

Basic action (integrals implied)

$$S = -4\alpha\eta^+\eta + \eta^+ \left[-\frac{\partial}{\partial t} + \nabla^2(ui + \alpha) \right] \xi + \xi^+ \left[-\frac{\partial}{\partial t} + \nabla^2(ui - \alpha) \right] \eta - \frac{ig_1}{2}\eta^+\xi^+\xi^2 - \frac{ig_2}{2}\xi^{+2}\eta\xi$$

with

$$g_1 = g_2 = g\frac{\hbar}{T_C}\left(\frac{\sqrt{2\pi}}{\lambda_T}\right)^d ,$$

where $\lambda_T = \sqrt{2\pi\hbar^2/mT_C}$ is the thermal de Broglie wavelength at the critical temperature T_C.

Effective potential

Expectation value of the field $\mathrm{Tr}\left[\hat{\rho}\hat{\varphi}(t,\mathbf{x})\right]$ time-independent due to translation invariance of GF.

Follow thermodynamics: construct **generating function of one-irreducible Green functions** Γ (Legendre transfrom of the generating function of connected Green functions $\log G(A)$ wrt source field)

$$\Gamma(\alpha) = \log G(A) - A\alpha\,,$$

where $A = A(\alpha)$ from

$$\alpha = \frac{\delta \log G(A)}{\delta A}\,.$$

This is ecpectation value of field operator in external field $\propto A$.

Physical quantities from Green functions II

Analysis reveals connection with action functional $S(\varphi)$ (notation!)

$$\Gamma(\varphi) = \frac{i}{\hbar}S(\varphi) + \text{sum of one-irreducible graphs of } \log G(\Delta^{-1}\varphi)\,.$$

Expectation value of the field operator $\varphi = \alpha$ is a function of time now.

Leading order of stationarity equation of variational solution is just the Gross-Pitaevskii for boson gas:

$$0 = \frac{\hbar}{i}\frac{\delta\Gamma}{\delta\varphi} = \frac{\delta S(\varphi)}{\delta\varphi} + \text{loop corrections}\,.$$

Renormalization of UV divergences

Graphs of QFT suffer from UV divergences.

Divergences are classified by large-momentum asymtotic behaviour of integrands of (one-irreducible) Feynman diagrams (which is usually power law, hence the name power-counting). Define **superficial degree of divergence**.

In a renormalizable model loop corrections to coupling constants in the effective potential are logarithmically divergent (degree of divergence is zero).

In modern calculations space dimension d is a complex parameter and UV divergences apppear as poles in $\varepsilon = d_c - d$, at $d = d_c$ log divergences: $Z = 1 + \dfrac{a_1(g,\varepsilon)}{\varepsilon} +$ poles of higher order.

Overview of renormalization group

Leading log corrections are colleted to redefinition (renormalization) of fields and parameters: $\varphi \to Z_\varphi \varphi$, $g \to Z_g g$.

Reference wavenumber μ in log is arbitrary: variation yields renormalization-group equations for renormalized Green functions (two-point function here):

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \gamma_\alpha \alpha \frac{\partial}{\partial\alpha} + 2\gamma(g)\right] G_R(\omega, \mathbf{k}, \alpha, \mu, g) = 0\,.$$

Coefficient functions β and γ determined by Z.

Critical scaling

Solution of RG equations connects values of Green functions at different values of parameters; at a fixed point of RG (node of the β function) scale-invariant behaviour:

$$G_R(\omega, \mathbf{k}, \alpha, \mu, g) = k^{-2+2\gamma_*} G_R\left(\omega\left(\frac{\mu}{k}\right)^{2-\gamma_\alpha^*}, \mu\left(\frac{\mathbf{k}}{k}\right), \alpha, 1, g_*\right).$$

If propagators of perturbation theory are generalized homogeneous functions of wave numbers and frequency, information of IR behaviour may be inferred from UV RG.

IR approximation of Keldysh function

The use of UV renormalization for large-scale analysis assumes certain homogeneity properties of propagators.

The retarded and advanced propagators possess the usual homogeneity of a parabolic differential operator.

In the Keldysh function put $\mu = 0$ and expand in $\mathbf{k}$:

$$\Delta^K(\omega, \mathbf{k}) = \left(1 + \frac{2}{\{\exp\left[\hbar\omega(\mathbf{k})/T_C\right]\} - 1}\right) \delta\left[\omega - \omega(\mathbf{k})\right] \sim \frac{2T_C}{\hbar\omega(\mathbf{k})} \delta\left[\omega - \omega(\mathbf{k})\right]$$

This expression with the desired homogeneity from $\omega(\mathbf{k}) = \hbar k^2/2m$ may be used for the UV power counting without regularization.

It is worth recalling that in the propagators of original field variables $\Delta + n_{DD}$ and $\widetilde{\Delta} + n_{DD}$ the desired homogeneity is absent.

Degree of divergence and canonical dimensions

Superficial degree of divergence of a one-irreducible graph:

$$\delta = d + 2 + (d-4)V_1 + (d-4)V_2 + dV_3 + dV_4 - \left(\frac{d+2}{2}\right) N_{\eta^+} - \left(\frac{d+2}{2}\right) N_{\eta} - \left(\frac{d-2}{2}\right) N_{\xi^+} - \left(\frac{d-2}{2}\right) N_{\xi}$$

V_i – number of vertices i, N_α – number of legs of the field α.

V_3 and V_4 have positive coefficients, the corresponding interaction structures are IR irrelevant and omitted.

Critical dimension of the effective IR model is $d_c = 4$.

Canonical dimensions of fields are $d_\xi = d_{\xi^+} = d/2 - 1$, $d_\eta = d_{\eta^+} = d/2 + 1$.

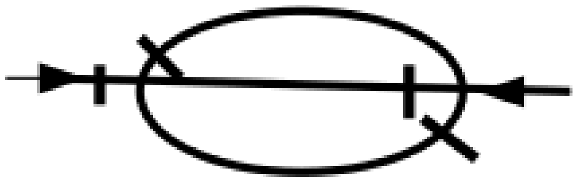
Choice of variables important: well-defined canonical dimensions in Keldysh variables. Symmetry between fields lost – this is a virtue, not drawback.

Regularization of the IR model

Effective Keldysh function regularized with constant γ

$$\Delta^{K}_{\mathrm{IR\ reg}}(\omega\,\mathbf{k}) = \frac{4\gamma T_C}{\hbar\omega(\mathbf{k})\left\{[\omega-\omega(\mathbf{k})]^2+\gamma^2\right\}}\,.$$

causes trouble. Consider two-loop contribution to $\Gamma_{\eta^+\eta}$:



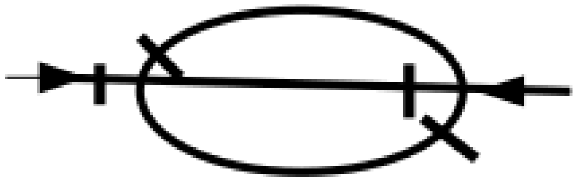
$$\propto \gamma\int\frac{d\mathbf{q}}{(2\pi)^d}\int\frac{d\mathbf{k}}{(2\pi)^d}\frac{1}{\omega(\mathbf{k})\omega(\mathbf{q})\omega(\mathbf{k}+\mathbf{q})}$$

$$\times\frac{1}{\left\{[\omega(\mathbf{k})+\omega(\mathbf{q})-\omega(\mathbf{k}+\mathbf{q})]^2+9\gamma^2\right\}}\,.$$

Divergence index of the unregularized graph is $\delta = 2(d-4)$. However, the divergence index of the wave-vector integral is $\delta = 2(d-5)$! Need limit $\gamma\to 0$ to restore the δ function. Very inconvenient, try something else.

Regularization with critical slowing down

Choose dispersion law consistent with homogeneity: $\gamma = a\omega(\mathbf{k}) \propto k^2$, $a > 0$ to obtain



$$\propto \int \frac{d\mathbf{q}}{(2\pi)^d} \int \frac{d\mathbf{k}}{(2\pi)^d} \frac{a}{\omega(\mathbf{k})\omega(\mathbf{q})\omega(\mathbf{k}+\mathbf{q})}$$

$$\times \frac{[\omega(\mathbf{k}) + \omega(\mathbf{q}) + \omega(\mathbf{k}+\mathbf{q})]}{\left\{[\omega(\mathbf{k}) + \omega(\mathbf{q}) - \omega(\mathbf{k}+\mathbf{q})]^2 + [\omega(\mathbf{k}) + \omega(\mathbf{q}) + \omega(\mathbf{k}+\mathbf{q})]^2 a^2\right\}} .$$

Now the homogeneity of the unregularized effective Keldysh function is restored together with degree of divergence and canonical dimensions.

Calculation of integrals a bit more complicated than in stochastic dynamics, but dealing with the parameter a may be postponed to the end.

Renormalized action of the effective model

Inspection of graphs shows that the effetive IR model is multiplicatively renormalizable with the renormalized action

$$S = -Z_0 \eta^+ \eta + \eta^+ \left(-Z_1 \frac{\partial}{\partial t} + Z_2 \nabla^2 \right) \xi$$

$$+ \xi^+ \left(-Z_3 \frac{\partial}{\partial t} + Z_4 \nabla^2 \right) \eta - \frac{1}{2} Z_5 \eta^+ \xi^+ \xi^2 - \frac{1}{2} Z_6 {\xi^+}^2 \eta \xi \,,$$

where the renormalization constants $Z_1, \ldots, Z_6$ are complex numbers. This is the reason of unorthodox notation here.

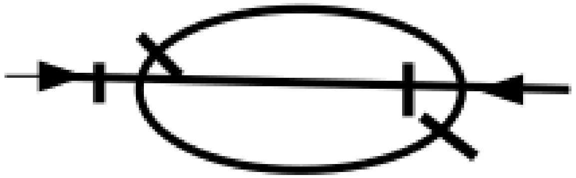
Symmetry arguments lead to constraints

$$Z_0(g_1, g_2) = Z_0^*(g_2^*, g_1^*), \qquad Z_1(g_1, g_2) = -Z_3^*(g_2^*, g_1^*),$$

$$Z_2(g_1, g_2) = -Z_4^*(g_2^*, g_1^*), \qquad Z_5(g_1, g_2) = -Z_6^*(g_2^*, g_1^*) \,.$$

Calculation of counterterms

The two-loop contribution to $\Sigma_{\eta^+\eta}$ is given by the familiar graph with a finite limit at $\alpha \to 0$ (dimensional regularization, $d = 4 - \varepsilon$)



$$= \int\limits_{q>m} \frac{d\mathbf{q}}{(2\pi)^d} \int \frac{d\mathbf{k}}{(2\pi)^d} \frac{16\, g_1 g_1^*}{k^2 q^2 (\mathbf{k}-\mathbf{q})^2}$$

$$\times \frac{\left[k^2 + (\mathbf{k}-\mathbf{q})^2 + q^2\right] \alpha}{\left\{\left[k^2 + (\mathbf{k}-\mathbf{q})^2 - q^2\right]^2 u^2 + \left[k^2 + (\mathbf{k}-\mathbf{q})^2 + q^2\right]^2 \alpha^2\right\}}$$

$$= \frac{g_1 g_1^*}{32\,\pi^4\,(u^2+\alpha^2)\,\varepsilon} \left\{ \pi u + 2u \arctan\left(\frac{u}{4\alpha} - \frac{3\alpha}{4u}\right) \right.$$

$$\left. + \alpha \log\left[\frac{4096\,\alpha^8}{(u^2+\alpha^2)\,(u^2+9\,\alpha^2)^3}\right]\right\} + O(1) \xrightarrow[\alpha\to 0]{} \frac{g_1 g_1^*}{16\,\pi^3\, u\, \varepsilon} + O(1)\,.$$

Generation of $\eta^+\eta$ term to action functional.

RG functions and critical dimensions

Three charges with rather complicated β functions ($\alpha = 1$ here):

$$\beta_{g1} = -\frac{2\left(2u^2+3\right)g_1 g_2 + u g_2^2 - u g_1^2}{8\pi^2\left(u^2+1\right)} - \epsilon g_1,$$

$$\beta_{g2} = \frac{2ug_1g_2 - \left(4u^2+5\right)g_2^2 + g_1^2}{8\pi^2\left(u^2+1\right)} - \epsilon g_2,$$

$$\beta_u = \frac{(u^2+6)ug_2^2 - ug_1^2 + 6g_1g_2}{64\pi^4(u^2+9)} - \frac{(u^2+1)}{128\pi^4}[(g_1^2+g_2^2)\mathrm{Im}M_2(u) + 2(g_1^2-g_2^2)\mathrm{Im}M_3(u) + 4g_1g_2\mathrm{Re}M_3(u)].$$

Just a single nontrivial IR stable fixed point with anomalous dimensions

$$\gamma_\xi^* = \frac{\epsilon^2}{100}\,,\ \gamma_\eta^* = \left(12\log\frac{4}{3} - 1\right)\frac{\epsilon^2}{100}\,,\ \gamma_\alpha^* = \left(1 - 6\log\frac{4}{3}\right)\frac{\epsilon^2}{50}\,,\ \omega = \frac{2\,\epsilon^2}{25}\ \log\frac{4}{3}\,.$$